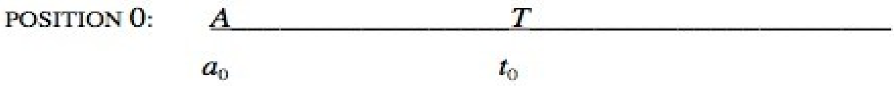
Subject 21

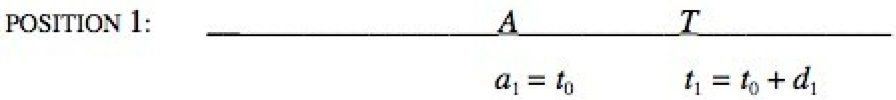
Please, don’t write on the exam paper.

Zeno’s paradox (from http://plato.stanford.edu/entries/zeno-elea/#Arr)

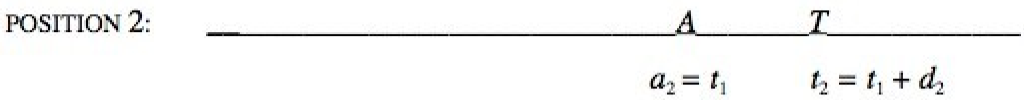
If a tortoise starts ahead of Achilles in a race, the tortoise will never be overtaken by Achilles. Let the start of the race be represented as follows:



During the time it takes Achilles to reach the point from which the tortoise started (t_0*), the tortoise will have progressed some distance (*d_1*) beyond that point, namely to* t_1*, as follows:*



Likewise, during the time it then takes Achilles to reach the new point the tortoise has reached (t_1*), the tortoise will have progressed some new distance (*d_2*) beyond the tortoise's new starting point, namely to* t_2*, as follows:*



The tortoise will again have progressed some further distance (d_3*) beyond* t_2*, namely to* t_3*, in the time it takes Achilles to move from* $a_2(=t_1)$ *to* $a_3(=t_2)$*. In fact, during the time it takes Achilles to reach the tortoise's location at the beginning of that time, the tortoise will always have moved some distance ahead, so that every time Achilles reaches the tortoise's new starting point, the tortoise will be ahead some. Therefore, the slowest runner in the race, the tortoise, will never be overtaken by the fastest runner, Achilles.*

Questions :

As in the text, we denote by d_n the distance between Achilles and the tortoise at step *n* but we do not consider the other variables such as a_n , $t_{n.}$ ..

We suppose that $d_0 = 100\,m$ and that the tortoise runs twice as slow as Achilles (The figure does not respect exactly this ratio)

1) Compute d_1 and prove that $d_2 = \frac{d_1}{2}$.

   In the same way, compute d_2 and prove that $d_3 = \frac{d_2}{2}$.

2) For any step *n* , what is the relation between d_n and d_{n+1} ?

3) What kind of sequence is (d_n) ? Write d_n in terms of *n*.

4) Prove that the total distance Achilles has ran at step *n* is $L_n = d_0 + d_1 + ... + d_n$.

5) *Let us recall that the sum of the n first terms of a geometric sequence with the initial term* u_0 *and the common ratio q (different from 1) is* $u_0 + u_1 + ... + u_n = u_0 * \frac{1-q^{n+1}}{1-q}$
Compute L_n in terms of *n*.

6) **Optional question**: Prove that the distance that Achilles has run is finite, and that he actually overtakes the tortoise in finite time.